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Collision risk model for independently operated homogeneous air traffic flows in terminal area

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Abstract

Collision risk models (CRM) have been developed and applied so as to determine the safe separation minima and monitor the collision risk level of the airspace. CRMs estimate the expected number of aircraft collisions. Each model employs some assumptions. For instance, the curved trajectories cannot be treated in some CRMs. These models cannot be generally applied for the collision risk estimation of the terminal airspace operation because the trajectories are curved in the terminal airspace. In addition, some CRMs cannot treat the time-dependent position error. The Rice CRM can handle time-dependent position errors and does not assume straight flight contrary to other CRMs. However, it estimates the expected number of collisions of a single pair of aircraft during a specific time interval though most of the other CRMs estimate the 'total' collision risk of all the aircraft in the airspace. The total number of possible collisions is often required rather than the expected number of collision of a single pair of aircraft in the application of collision risk models.

Our motivation is to extend the Rice formula so as to calculate the 'total' collision risk. We developed a CRM for the 'total' collision risk even in the curved-trajectory and time-dependent position error cases. The assumption we made for the derivation of our CRM formula was 'independently operated homogeneous air traffic flow.' The Rice CRM was used as the basis of this CRM. We also applied our CRM to the collision risk estimation of independently operated arrival and departure flows.

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1. Introduction

In air traffic control, separation minima[1] are determined in order to prevent collision occurrences. Air traffic controllers should control traffic without violating the separation minima. Safety assessments should be conducted so as to find the appropriate value of the separation minimum. A typical safety study procedure is, first identifying hazards[1] and secondly estimating the frequency and severity of unsafe events. Collision risk models (CRMs) are used for the precise and dependable estimation of the frequency of collision occurrences.

Several CRMs are now employed for collision risk estimation. However, these CRMs are limited in application. For instance, some CRM cannot treat time-dependent position error. Most CRM cannot estimate risk of collisions if flight route is curved. Recently, congestion in terminal area becomes more serious. The safety analysis in the complex flight environment (straight flights, curved and descending/ascending flights) as in terminal area is indispensable. Only Rice CRM can handle time-dependent position errors and does not assume straight flight, but it only considers the expected number of collision occurrence of a given single pair of aircraft. We often want to estimate the expected number of total collisions during a long time period in a given airspace rather than the collision risk of a single pair of aircraft.

The expected number of total collisions is equivalent to the sum of collision risk with respect to all the possible aircraft pairs. In the practice of collision risk estimation, we do not consider all the possible pairs, but consider the pairs satisfying the given conditions. These conditions depend on the separation minima in consideration. For instance, when we consider route spacing, the additional condition will be that the aircraft pairs fly on the adjacent routes at the same flight level. We say that a CRM considers 'total risk' if it explicitly considers the sum of collision risk with respect to all the possible aircraft pairs satisfying the given condition[2]. The Rice CRM does not consider the 'total risk.'

We propose a new CRM which estimates the 'total risk' of complex flights by expanding an existing CRM. We employed assumptions on the characteristics of traffic flow in the course of derivation of the new CRM. The first assumption is independent operation, and the second is the 'homogeneity' defined later.

This paper is organized as follows. In the second section, features of current CRMs are compared with the feature of the proposed CRM in detail so as to clarify the difference of the proposed CRM with the current CRM. Definition and formulation with risk exposure function for the proposed CRM has been shown in the third section. Fourth section is devoted to application examples of the new CRM. It will demonstrate the validity of this new CRM. We conclude this paper in the fifth section.

2. Features of current CRMs

We review the features of four major CRMs, namely, Reich CRM, Hsu CRM, Anderson CRM and Rice CRMs first. They have limitations. Thus, we propose a new CRM for independently operated homogeneous traffic flows in the next chapter. It overcomes the limitations of the above CRMs. We give a comparison of the new CRM with other CRMs in this section.

2.1. Reich CRM

The oldest well-known CRM is Reich CRM[2]. The Reich CRM was developed for the estimation of collision risk of

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1 In this paper, separation minimum does not only mean separation minimum in the ordinary sense such as longitudinal/vertical separation minimum, radar separation etc. It also means the criteria on the separation of aircraft such as route spacing and minimum separation between terminal flight procedures.

2 The risk metric employed by a CRM which considers 'total risk' should not necessarily be the expected number of collisions. For instance, the expected number of collisions per flight hour is sometimes used as the metric of collision risk. It is the quotient of expected number of collisions divided by total flight hour, and is easily obtained from the expected number of collisions.

long-range air traffic systems such as trans-ocean traffic. The Reich CRM assumes the time-independent position errors. In addition, aircraft are assumed to fly straight. The Reich CRM gives the 'total risk' estimation.

The Reich CRM has been applied to many problems. It is used for the collision risk estimation of reduced vertical separation minima (RVSM)[3],[4] and that of typical route spacing including Performance-Based Navigation (PBN)[5] separation. Although the Reich CRM has been applied in many operational scenarios, its application is mainly limited to the risk estimation of parallel route separation and longitudinal/vertical separation of aircraft flying straight.

For instance, we cannot use this CRM for the collision risk estimation of the distance-based longitudinal separation under the ADS-C (Automatic Dependent Surveillance)[6],[7]. The current position and speed (or equivalent information) is transmitted from aircraft to the ATC authority under the ADS-C. The reporting time interval is longer than ten minutes in general. The ATC authority makes an estimation of the aircraft position using this information until the next position report is received. The magnitude of estimation error of the current position is roughly calculated by (the measurement error reported in the position report) + (the error in the speed information) × (time elapsed from the previous report transmission). It is not time-independent any more.

2.2. Hsu CRM

Hsu[8] developed another CRM, called the Hsu CRM in this paper, applicable to track-crossing cases. It can handle the time-dependent position error case unlike the Reich CRM. However, the tracks of aircraft are assumed to be straight even in this model. It does not consider the 'total risk,' neither.

2.3. Anderson CRM

Anderson et al.[9] refined the Hsu CRM. We call it the Anderson CRM. It considers the total risk during a given time interval in the similar way as the Reich CRM. The idea was as follows: The risk of collision for a pair of aircraft is negligible if they are extremely separated. They took into account the possibility of the event that two crossing aircraft are proximate (judging from information available to air traffic controllers). The risk of collision is the product of the proximity probability with the collision risk of proximate pairs.

The assumption that aircraft flies straight is not so restrictive when we consider separation minima and route spacing in oceanic airspaces and en-route airspaces. However, it is not the case for flights in terminal airspaces. Aircraft make turns frequently in terminal airspaces. Drift caused by unexpected wind during the turns should also be modelled. It may be possible to assume a simplified trajectory model which is piecewise linear in these situations. However, it seems too simplistic.

2.4. Rice CRM

Mehadhebi et al.[10],[11]developed another CRM applicable to more general scenarios. It is called the Rice CRM named after Rice formula, which is the basis of this CRM. The Rice CRM is applicable even in time-dependent position error cases and even in cases where the tracks of aircraft are curving. Therefore, we can use this CRM for the collision risk estimation of the distance-based longitudinal separation under the ADS-C, and for the collision risk estimation of operations in terminal airspaces. In fact, the Reich, Hsu and Anderson CRMs can be derived from the Rice CRM as in Reference [10].

The Rice CRM considers the expected number of collisions for a pair of aircraft or that of an obstacle and an aircraft in a time interval. The theory of 'total risk' is not included in the fundamental formula of the Rice CRM. When we develop a safe separation between flight procedures, we have interest in the total risk of many aircraft following the given flight procedures rather than the collision risk of a pair of aircraft. We need some derivation so as to obtain a CRM considering 'total risk.'

2.5. Proposed CRM

We propose a CRM of two independently-operated homogeneous traffic flows in this paper. It is based on the theory of the Rice CRM, and takes on good features of the Rice CRM. In addition, it enables the calculation of the total risk during a given time interval. Table 1 summarizes the main features of CRMs including the CRM we propose in this paper.

Table 1: Main feature of CRMs

CRM	**Can be used in time-dependent position error case**	**Can be used in curved trajectory cases**	**'Total risk' is considered**
Reich	No	No	Yes
Hsu	Yes	No	No
Anderson	Yes	No	Yes
Rice	Yes	Yes	No
Proposed	Yes	Yes	Yes

3. Proposed CRM for independently-operated homogenous traffic flows

We propose a new CRM for two independently-operated homogeneous traffic flows during a time interval in this section. It is based on the theory of Rice CRM, and inherits good features of it.

3.1. Notations and Rice CRM

We consider a pair of aircraft in this subsection. We can apply the same idea for the case of a pair of aircraft and an obstacle. Aircraft have some physical volume and a collision between them occurs if and only if their volumes overlap spatially. When we consider the relative motion, we can assume that one aircraft is moving and the other stays at the origin[3].

A *collision volume* Ω is a volume such that the collision of a particle (a representative point of the aircraft) with it is equivalent to the collision of two aircraft in consideration. When we simplify the shape of aircraft, we can easily define the collision volume. Two kinds of collision volumes have been used, the box-shaped collision volume for the estimation of collision risk of the same/opposite-direction traffic, and the cylinder-shape for the crossing-track traffic. When two aircraft are box-shaped and their size is identical, two boxes (B1 and B2) experience the collision if and only if the particle located at the center of B1 experiences the collision with the double-sized box at the center of B2. In this case, the collision volume is the double-sized box.

Let $\mathrm{X}_{rel} = (\mathrm{x}_{rel}, \mathrm{y}_{rel}, \mathrm{z}_{rel})$ be the position of the particle. Also, let $\mathrm{V}_{rel} = (\dot{x}_{rel}, \dot{y}_{rel}, \dot{z}_{rel})$ be its speed. In collision risk estimation, these variables are considered to be random variables. Let $\mathrm{f}_{rel}(\mathrm{t}, \mathrm{X}_{rel}, \mathrm{V}_{rel}) = \mathrm{f}_{rel}(\mathrm{t}, \mathrm{x}_{rel}, \mathrm{y}_{rel}, \mathrm{z}_{rel}, \dot{x}_{rel}, \dot{y}_{rel}, \dot{z}_{rel})$ be their probability density function at time t. According to

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[3] Aircraft fly in the three dimensional space, but we assume that aircraft moves in the one dimension for simplicity. Let $x_1(t)$ and $x_2(t)$ be the nominal positions (the aircraft positions when the aircraft perfectly follows the published procedure without any position/timing error). Let $\varepsilon_1(t)$ and $\varepsilon_2(t)$ be the position errors. The actual positions of both aircraft are $x_1(t) + \varepsilon_1(t)$ and $x_2(t) + \varepsilon_2(t)$, respectively.
When we consider the relative position, the nominal position of the first aircraft is at the origin, and we assume that there is no position error for the first aircraft. On the other hand, the nominal position of the second aircraft is $x_2(t) - x_1(t)$, and the position error is $\varepsilon_2(t) - \varepsilon_1(t)$. The collision occurs in the first case if and only if so does in the second case. We can consider the speed component in the same way.

the Rice CRM, the expected number of occurrences of that the particle enters into the collision volume during the time period $[t_0, t_1]$ is given by the following formula:

$$CR_{single\ pair} = \int_{t_0}^{t_1} \Psi(\mathrm{u})\mathrm{du} \tag{1}$$

Here, the notation Ψ denotes the risk exposure function, and defined by the following equation:

$$\Psi(\mathrm{t}) = \int_{\partial\Omega}\int_{\mathbb{R}^3} \mathrm{f}(t, X_{rel}, V_{rel})(n \cdot \mathrm{V}_{rel})^+ dV_{rel}\, dX_{rel} \tag{2}$$

The notation $\partial\Omega$ is the boundary of the collision volume and the vector n is the normal vector of the surface $\partial\Omega$ directed to the interior of the collision volume. $n \cdot \mathrm{V}_{rel}$ is the inner product and $(n \cdot \mathrm{V}_{rel})^+$ is given by the following:

$$(n \cdot \mathrm{V}_{rel})^+ = \begin{cases} n \cdot \mathrm{V}_{rel} & if\ n \cdot \mathrm{V}_{rel} > 0 \\ 0 & otherwise \end{cases} \tag{3}$$

3.2. Definition of a homogeneous traffic flow

We derive the formula for the collision risk of two independently-operated homogeneous traffic flows in this chapter.

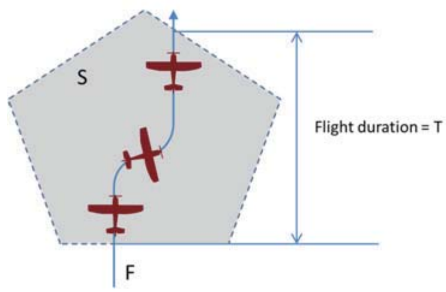

Figure 1: Air traffic flow

We first define terminologies to present our theorem. Let us consider the air traffic flow to be F and the airspace to be S. Figure 1 illustrates a snapshot of a traffic flow and an airspace S. The traffic flow F represented by a curved arrow goes through airspace S. Three aircraft are in the flow F in the figure. Air traffic flow F is homogeneous (in size, density, trajectory and position error) in airspace S if the following conditions are satisfied.

(i) The sizes of all aircraft in F are identical.

(ii) The time (T) required to cross airspace S is identical for all aircraft in flow F. We call T travelling time duration.

(iii) The joint distribution of the position and speed of all aircraft is identical. Consider two arbitrary aircraft A and B in flow F. Let t_A and t_B be the time when the aircraft A and B entered airspace S, respectively. (We assume that any aircraft in F enters airspace S only once.) The joint distribution of the position and speed of aircraft A at the time $t + t_A$ is identical to that of the aircraft at the time $t + t_B$ for any $0 \le t \le T$. Here, T denotes the travelling time duration in the considered airspace S. We call it the distributions of the position and speed of flow F at the elapsed time t.

(iv) The flow rate of F is time-independent.

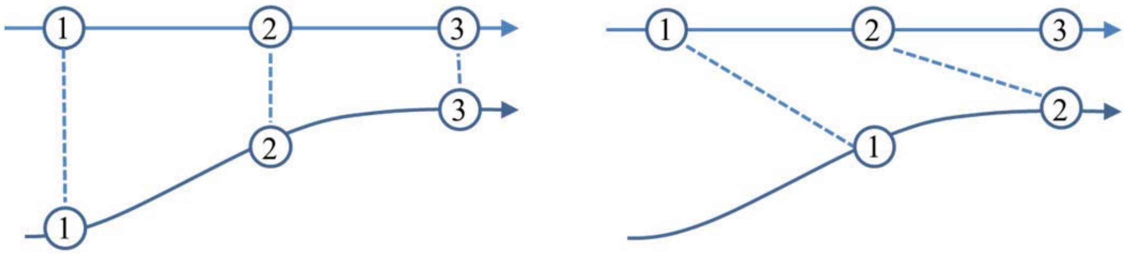

Figure 2: Two feasible trajectories of independently-operated traffic flow

For instance, let us consider a departure procedure from an airport. If the procedure is well-defined and aircraft follow the procedure, we can expect that the trajectories of these aircraft will seem almost identical. Aircraft following the procedure create a traffic flow.

In the Reich, Hsu and Anderson CRMs, a 'typical' aircraft is often considered. The aircraft considered in these CRMs are assumed to be all typical aircraft. Typical aircraft are of identical size (the average size of aircraft in the considered airspace is often used) and identical navigation performance. In this sense, the first three conditions of the definition of a homogeneous flow are as restrictive as these CRMs. The last condition seems a little restrictive and unrealistic. However, we may use the value of the flow rate at rush hour for conservative assessment of the risk. Or, we can calculate the risk both for rush hours and normal hours, and calculate the weighted average of them with respect to flight time.

3.3. CRM for independently operated homogeneous air traffic flows

We consider two independently-operated traffic flows. Note that the homogeneity defined above is the characteristic of a single flow, but independently-operation is the relationship of multiple flows. In the independently-operated traffic flows, the operation in any flow is conducted without being influenced by the state of the other flows. The solid lines in Figure 2 illustrate air traffic flows. The circled numbers are the estimated positions of given aircraft. If the number in the figure is identical, they represent the positions of two aircraft at the same time. The dotted lines represent the estimated separation of aircraft. The estimated separation on the left side of the figure is shorter than that on the right side. The collision risk of the two aircraft on the left seems much higher than that on the right. The chance of occurrence of the left situation is identical with that on the right. In this situation, assuming a specific trajectory, such as the left side of the figure, may lead to an overestimation or an underestimation of the risk. Therefore, we should consider the total risk taking into account of both situations shown in the left and right figures. The following is the main theorem of this paper.

Theorem

Let us consider an airspace S and two independently operated air traffic flows F_1 and F_2 which are homogeneous in airspace S. The flow rates of air traffic flows F_1 and F_2 are n_1 (aircraft per hour) and n_2 (aircraft per hour), respectively. The notations T_1 and T_2 are the travelling time durations of flows F_1 and F_2 in S, respectively.

Let $f_1(t, X, V)$ and $f_2(t, X, V)$ be the probability density function of the distribution of position X and speed V of single aircraft in the flows F_1 and F_2 at the elapsed time t, respectively. They are independent from each other. Then, the expected number of collisions of aircraft in F_1 with aircraft in F_2 in airspace S is approximated by

$$CR_{multiple\ pairs} = n_1 n_2 \int_0^{T_1} \int_0^{T_2} \Psi(\mathrm{s}, \mathrm{t})\, dt\, ds \quad \text{(collisions per hour)} \qquad (4)$$

where Ψ(s, t) is a risk exposure function defined by the following equations. The *s* and *t* are the elapsed times since aircraft enter the flows into the flows F_1 and F_2, respectively.

$$f_{rel}(s, t, X_{rel}, V_{rel}) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_1(s, X_{rel}+\zeta, V_{rel}+\eta) f_2(t, \zeta, \eta) d\zeta\, d\eta \tag{5}$$

$$\Psi(t) = \int_{\partial\Omega}\int_{\mathbb{R}^3} f(s, t, X_{rel}, V_{rel})(n \cdot V_{rel})^{+} dV_{rel}\, dX_{rel} \tag{6}$$

Proof.

We may assume that $T_1 \le T_2$ without loss of generality. We first fix arbitrary aircraft A and B in the flow F_1 and F_2, respectively. We can evaluate the collision risk for a given pair of aircraft by means of the original Rice CRM. Let $t_{A,ini}$ and $t_{B,ini}$ be the entering time of aircraft A and B into airspace S, respectively. If two aircraft are not in airspace S at the same time, we consider that there is no risk of collision in airspace S. If we take into consideration this point, we can calculate the expected number of collisions of aircraft A with aircraft B by the original Rice formula. It is given by the following formulae.

$$h(v) = \begin{cases} 0 & if\ T_2 \le v \\ \int_0^{T_2-v} \Psi(u, u+v) du & if\ T_2 - T_1 \le v \le T_2 \\ \int_0^{T_1} \Psi(u, u+v) du & if\ 0 \le v \le T_2 - T_1 \\ \int_{-v}^{T_1} \Psi(u, u+v) du & if\ -T_1 \le v \le 0 \\ 0 & if\ v \le -T_1 \end{cases} \tag{7}$$

Here, $v = t_{A,ini} - t_{B,ini}$. Note that equation (7) holds true for an arbitrary pair of aircraft A in F_1 and B in F_2 because of the conditions (i), (ii) and (iii) of the definition of homogeneous air traffic flows.

We next consider the expected number C_A of collisions of a given aircraft A in F_1 with an arbitrary aircraft in F_2. The number of aircraft in the flow F_2 satisfying the condition $p \le v \le p + \Delta t$ is approximately $n_2\Delta t$ for any time p and $\Delta t > 0$ (The condition (iv) of the definition of homogeneous traffic flows is used). Hence, the expected number of collisions of aircraft A with an aircraft in F_2 satisfying the condition $p \le v \le p + \Delta t$ is approximately $n_2 h(p)\,\Delta t$. C_A is calculated by the following equation.

$$C_A = n_2 \int_{-\infty}^{+\infty} h(v) dv = n_2 \int_0^{T_1}\int_0^{T_2} \Psi(s, t) dt\, ds \tag{8}$$

Finally, again by the condition (iv) of the definition of homogeneous traffic flows, the expected number of collisions of aircraft in F_1 with aircraft in F_2 in airspace S is given by equation (4).

The proposed CRM evaluates the expected number of collisions per hour. However, it is easy to calculate the total flight hours during the given time period when the flow rates and the travelling duration of time are given. If we count a collision as two accidents, the collision risk measured in fatal accidents per flight hour is calculated by the following formula.

$$CR_{faf} = \frac{2CR_{multiple\ pairs}}{n_1T_1+n_2T_2} = \frac{2n_1n_2\int_0^{T_1}\int_0^{T_2}\Psi(s,t)dtds}{n_1T_1+n_2T_2} \tag{9}$$

(fatal accidents per flight hour)

3.4. Simplified risk exposure function

The collision volume is given as a box or a cylinder in the Reich, Hsu and Anderson CRMs. We will give the simplified forms of the risk exposure functions in the case where the collision volume is a box and a cylinder. A box-shaped collision volume is used for parallel air traffic flows and cylindrical collision volume is used in other cases.

Proposition 1

Let us consider an airspace S and two independently-operated parallel air traffic flows F_1 and F_2 which are homogeneous in airspace S. The risk exposure function is approximated by

$$\Psi(\mathrm{s},\mathrm{t}) = \mathrm{P}_{x,s,t}(overlap)\mathrm{P}_{y,s,t}(overlap)\mathrm{P}_{z,s,t}(overlap)\left(\frac{\overline{|\dot{x}_{rel}|}_{s,t}}{2\lambda_x} + \frac{\overline{|\dot{y}_{rel}|}_{s,t}}{2\lambda_y} + \frac{\overline{|\dot{z}_{rel}|}_{s,t}}{2\lambda_z}\right) \tag{10}$$

if the following assumptions hold true.

- The collision volume is a box. The length, width and height of it are $2\lambda_x$, $2\lambda_y$ and $2\lambda_z$, respectively. In other word, the parameters λ_x, λ_y and λ_z denote the distance from the center of the volume to the boundary in each dimension. The length direction, width direction and height direction are parallel to the x-axis, y-axis and z-axis, respectively.

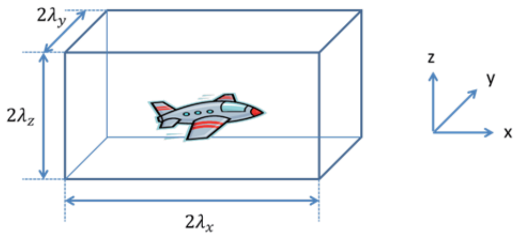

Figure 3: Box-shaped collision volume

- The distributions of position and speed of aircraft in x,y,z-directions are mutually independent in the following sense. There exist continuous probability density functions $f_{1,x}(\mathrm{t},\mathrm{x},\dot{x})$, $f_{1,y}(\mathrm{t},\mathrm{y},\dot{y})$, $f_{1,z}(\mathrm{t},\mathrm{z},\dot{z})$, $f_{2,x}(\mathrm{t},\mathrm{x},\dot{x})$, $f_{2,y}(\mathrm{t},\mathrm{y},\dot{y})$ and $f_{2,z}(\mathrm{t},\mathrm{z},\dot{z})$ such that $f_i(t,X,V) = f_{i,x}(\mathrm{t},\mathrm{x},\dot{x})f_{i,y}(\mathrm{t},\mathrm{y},\dot{y})f_{i,z}(\mathrm{t},\mathrm{z},\dot{z})$ for i=1,2.
- Consider the restriction of the probability density function of the distribution of relative position and relative speed $f_{rel}(s,t,X_{rel},V_{rel})$ in the equation (5). We assume that it is approximately constant on a neighbourhood of the collision volume. (It is approximated by $f_{rel}(s,t,0,V_{rel})$ on a neighbourhood of $\mathbb{R}\times\mathbb{R}\times[-\lambda_x,\lambda_x]\times[-\lambda_y,\lambda_y]\times[-\lambda_z,\lambda_z]\times\mathbb{R}^3$.)

Here, $P_{x,s,t}$(overlap) is the probability that a pair of aircraft in air traffic flows F_1 and F_2 overlap in the x-direction under the condition that the aircraft in F_1 and F_2 travelled s hours and t hours, respectively, since they entered into airspace S. The notation $\overline{|\dot{x}_{rel}|}_{s,t}$ is the average relative longitudinal velocity of a pair of colliding aircraft in air traffic flows F_1 and F_2 under the same condition as above. We define the other parameters in the same way.

Proof. Proof is provided in Appendix A.

The speed component is considered to be constant or independent of the position distribution in the Reich, Hsu and Anderson CRMs. In our simplification of the risk exposure function, the assumption on the correlation between position and speed uncertainties are weakened as in the last condition of Proposition 1. We can assume that the last condition is satisfied when the size of aircraft λ_x, λ_y and λ_z are significantly smaller than the scale parameters of distribution models.

Proposition 2

Let us consider an airspace S and two independently-operated air traffic flows F_1 and F_2 which are homogeneous in airspace S. The risk exposure function is approximated by

$$\Psi(\mathrm{s},\mathrm{t}) = \mathrm{P}_{hor,s,t}(overlap)\mathrm{P}_{z,s,t}(overlap)\left(\frac{2\overline{|v_{hor,rel}|}_{s,t}}{\pi\lambda_h} + \frac{\overline{|\dot{z}_{rel}|}_{s,t}}{2\lambda_z}\right) \quad (11)$$

if the following assumptions hold true.

- The collision volume is a cylinder. The radius and height of it are $2\lambda_h$ and $2\lambda_z$, respectively. In other word, the parameters λ_h and λ_z denote the distance from the center of the volume to the boundary in the horizontal and vertical dimensions, respectively. The height direction is parallel to the z-axis.

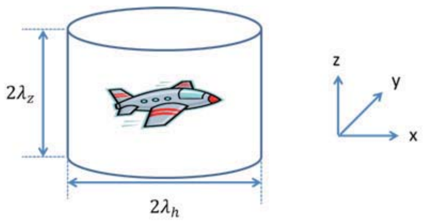

Figure 4: Cylinder-shaped collision volume

- The distributions of position and speed in a horizontal plane and z-direction are mutually independent in the following sense. There exist continuous probability density functions $f_{1,h}(\mathrm{t},\mathrm{x},\dot{x},y,\dot{y})$, $f_{1,z}(\mathrm{t},\mathrm{z},\dot{z})$, $f_{2,h}(\mathrm{t},\mathrm{x},\dot{x},y,\dot{y})$ and $f_{2,z}(\mathrm{t},\mathrm{z},\dot{z})$ such that $f_i(t,X,V) = f_{i,h}(\mathrm{t},\mathrm{x},\dot{x},\mathrm{y},\dot{y})f_{i,z}(\mathrm{t},\mathrm{z},\dot{z})$ for i=1,2.
- Consider the restriction of the probability density function of the distribution of relative position and relative speed $f_{rel}(s,t,X_{rel},V_{rel})$ in the equation (5). We assume that it is approximately constant on a neighbourhood of the collision volume. (It is approximated by $f_{rel}(s,t,0,V_{rel})$ on a neighbourhood of $\mathbb{R}\times\mathbb{R}\times\mathrm{D}(0,\lambda_h)\times[-\lambda_z,\lambda_z]\times\mathbb{R}^3$. Here, $\mathrm{D}(0,\lambda_h)$ denotes the 2-dimesional closed disc centred at the origin with radius of λ_h.)

Here, $P_{hor,s,t}$(overlap) is the probability that a pair of aircraft in air traffic flows F_1 and F_2 overlap in the horizontal dimension under the same condition as Proposition 1. The notation $\overline{|v_{hor,rel}|}_{s,t}$ is the average relative horizontal velocity of a pair of colliding aircraft in air traffic flows F_1 and F_2 under the same condition as above.

Proof. Proof is similar to that of Proposition 1, so we omit the proof.

4. Application example

In this section, we apply our CRM to an example scenario. Section 4.1 introduces the example scenario whose collision risk is assessed. Aircraft kinematic model and wind model are discussed in Section 4.2. The detailed aircraft trajectories and their uncertainty models are given in Section 4.3. Section 4.4 summarizes the other assumptions used for the estimation of collision risk. It is not easy to calculate the horizontal overlap probability, which is a parameter of collision risk model. We describe how we computed the horizontal overlap probability in Section 4.5. Finally, Section 4.6 summarizes the result.

4.1. Example scenario

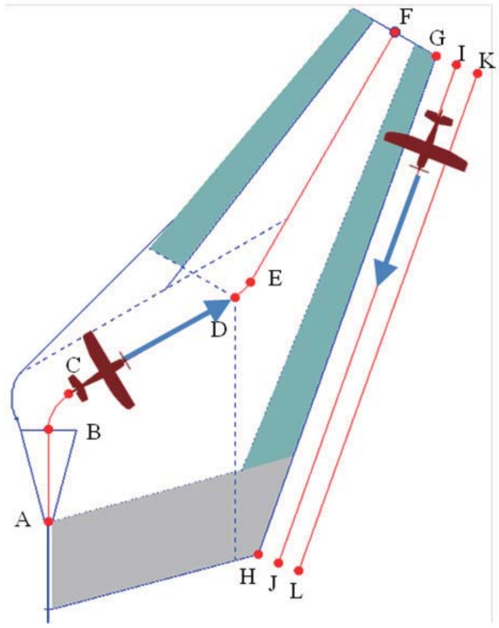

Figure 5: Geometry of scenario

Table 2: Coordinates of points in nautical miles

Category (Assumption: arrival and departure aircraft belonging to an identical aircraft category)	Cat. A		Cat. C	
RNP-type (departure)	RNP 0.3	RNP 1	RNP 0.3	RNP 1
A	(0.00,0.00)		(0.00,0.00)	
B	(0.00,1.92)		(0.00,1.92)	
C	(0.42,2.65)		(2.02,5.41)	
D	(3.89,4.65)		(5.48,7.41)	
E	(4.21,4.97)		(6.99,8.92)	
F	(7.21,10.16)		(9.99,14.12)	
G	(8.07,9.66)		(10.86,13.62)	
H	(3.78,–0.84)		(5.10, –0.49)	
I	(8.50,9.49)	(9.49,9.08)	(11.71,13.27)	(12.28,13.04)
J	(4.21,–1.02)	(5.20,–1.42)	(5.53,–0.66)	(6.52,–1.07)
K	(8.92,9.32)	(10.91,8.51)	(11.29,13.45)	(13.69,12.46)
L	(4.63,–1.19)	(6.62,–2.00)	(5.95,–0.84)	(7.94,–1.65)

We consider the case where one aircraft is flying on VOR SID and the other is on RNP STAR. Figure 5 illustrates the nominal horizontal trajectory of two aircraft. The word 'nominal trajectory' means the trajectory followed by aircraft when aircraft have the zero position error. A departing aircraft takes off at the point A and make a 60 degrees turn at the point B at the height of 400 feet. Aircraft flies straight 4 NM after the turn (Segment CD), then make a 30 degrees turn to the left. It flies straight to the VOR position F and passes over F at the altitude of 4,000 feet. On the other hand, an arriving aircraft is directing to the point I and make a 7.8 degrees fly-by turn to the left at the point I. After that, it directs to the point J and makes a 22.2 degrees fly-by turn to the left at the point J. We consider two aircraft category types defined in PANS-OPS[12] and RNP types for collision risk evaluation. The coordinates of the points are not identical in all cases. They are defined in Table 2. Cat. A and Cat. C in the table represents Category A and C aircraft defined in PANS-OPS, respectively.

4.2. Aircraft kinematic model and wind model

We will first develop a nominal trajectory model of aircraft. The following kinematic equations hold true under the assumption that the aircraft is a point of mass.

$$\begin{cases} N\cos\alpha = mg \\ N\sin\alpha = mr\omega^2 \end{cases} \tag{12}$$

Here, N is lift, g is the gravity constant, m is the mass of the aircraft, r is the radius of turn, ω is the angular rate, and α is the bank angle. On the other hand, the following formula holds true.

$$v = r\omega \tag{13}$$

We get the following formula solving the above equations.

$$\omega = \frac{g\tan\alpha}{v} \tag{14}$$

Converting the units, we get the following formula:

$$R(\deg/\sec) = \frac{180}{\pi}\omega(rad/\sec) = \frac{180}{\pi}\frac{g(NM/\sec^2)\tan\alpha}{v(NM/\sec)} = \frac{3431\tan\alpha}{\pi V} \tag{15}$$

We postulated that the gravity constant g is 68625 NM/hour^2. We get the radius of turn as follows:

$$r(NM) = \frac{v(NM/\sec)}{\omega(rad/\sec)} = \frac{V(knots)/3600}{\frac{\pi}{180}R(\deg/\sec)} = \frac{V}{20\pi R} \tag{16}$$

We will take the wind effect into account in the turn trajectory model. (See Figure 6.) The cumulative distance an aircraft drifts due to the wind effect is given by (the wind speed) × (the time elapsed from the initiation of the turn). The wind speed is given by w/3600 (NM/sec). On the other hand, the time elapsed from the initiation of the turn is (θ/R), where θ is turn angle. Hence, the aircraft is expected to be within the circle of the radius $E_\theta = (\theta/R)\times(w/3600)$ centered at the position of the aircraft in the no-wind case. To be conservative, we assume that the aircraft is on the circle. We also assume a cross wind. Then the wind vector is orthogonal to the aircraft speed vector. Hence, the ground speed should be parallel to the bold arrow in the figure. Therefore, the aircraft current position b2 is the point of the circle whose tangent is parallel to the bold arrow. Hence the angle made by $\overrightarrow{bb_1}$ and $\overrightarrow{bb_2}$ is $\arctan(w/V)$.

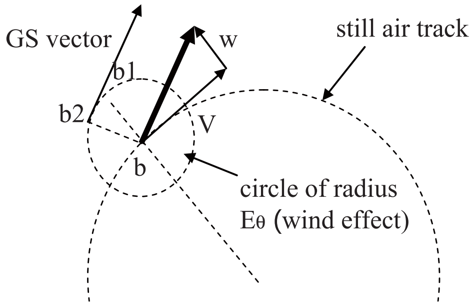

Figure 6: Wind effect

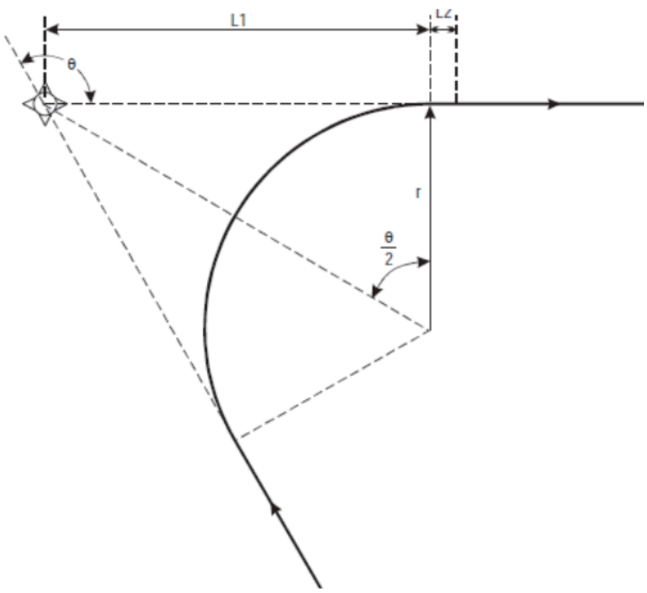

Figure 7: Fly-by turn model[12, Figure III-2-1-8]

The wind effect makes a large turn radius. As a result, the wind effect gives a conservative estimation of collision risk if the wind brings the turning aircraft closer to the other aircraft. Alternatively, the wind effect may make underestimation of collision risk if the wind brings the turning aircraft farer to the other aircraft. In this case, in general, the assumption of instantaneous turn makes for a conservative assumption. Hence, we take the wind effect into account in the second turn of a departing aircraft and the turns of an RNP aircraft, but the initial turn of the departing aircraft is assumed to be instantaneous.

The fly-by trajectory model are specified in PANS-OPS[12, Figure III-2-1-8]. We will utilize this fly-by trajectory model for the nominal trajectory model of the fly-by turn for arriving aircraft. (See Figure 7.)

4.3. Trajectory and its uncertainty modelling

Nominal Trajectory/Uncertainty Model for Departing Aircraft

We define the nominal trajectory and uncertainty model for departing aircraft. The following trajectory is assumed.

- IAS (Indicated Air Speed) = 121 knots for Cat. A aircraft and IAS = 264 knots for Cat. C aircraft during the flight. We utilize the following conversion formula for converting from IAS to TAS (True Air Speed).

$$\mathrm{TAS} = \mathrm{IAS} \times 171233\,[(288 \pm \mathrm{VAR}) - 0.00198H]^{0.5} \div (288 - 0.00198H)^{2.628} \quad (17)$$

where VAR = temperature variation about ISA (international Standard Atmosphere)　in °C, H = altitude in feet. VAR is assumed to be 15.

- Taking off at the runway threshold at time = 0 with heading = 0.
- Climbing straight till 400 ft with the gradient = 3.3%. TAS is assumed to be identical to the TAS at 400ft during the climb. No wind is assumed.
- Make a 60° right turn at 400ft. No wind is assumed. TAS is assumed to be identical to the TAS at 400ft during the turn.
- 4NM straight flight. TAS is assumed to be identical to the TAS at 1,200ft during the flight. No wind is assumed.
- Make a 30° left turn at 1,200ft. The 30 knots cross-wind is assumed. TAS is assumed to be identical to the TAS at 1,200ft during the turn.
- 6NM straight flight. TAS is assumed to be identical to the TAS at 4,000ft during the flight. No wind is assumed.
- Departing aircraft are assumed to climb at a constant gradient. The gradient is determined for departing aircraft to reach 4,000 ft over VOR. For Category A aircraft, a 5.0% gradient is assumed. For Category C aircraft, a 3.6% gradient is assumed. (See Figure 8.)
- **(Position Uncertainty of Turn Initiation Area)** Assume that the PANS-OPS area is defined by a 2 sigma value of position uncertainty of aircraft (Gaussian distribution is assumed). Namely, twice of the standard deviation of position uncertainty is defined by the following formula.

$$150(\mathrm{m}) + \tan(15°)\times(\text{distance from DER in meters}) \quad (18)$$

For conservative assessment, we assume that the along/cross-track uncertainty is assumed to follow a double exponential distribution (DE). It is also assumed that the 95% containment probability of the Gaussian distribution model is identical to that of DE model. Under this assumption, the scale parameter λ of the DE model is given by the following formula.

$$\lambda = 1.96\times(150/1852(\mathrm{NM}) + \tan(15°)\times(\text{distance from DER in NM}))/3$$
$$=0.053+0.175\times(\text{distance from DER in NM})$$
$$=0.389 \quad \text{(at first turn initiation point)} \quad (19)$$

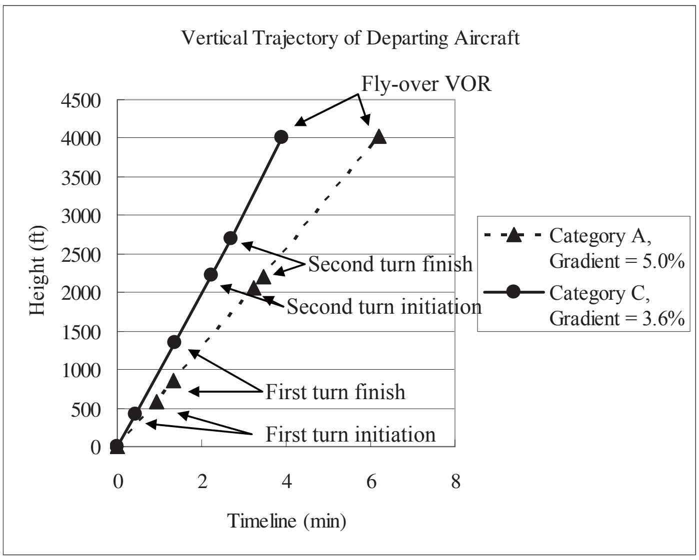

Figure 8: Nominal vertical trajectory

- **(Position Uncertainty before Second Turn Initiation)** Along-track error and cross-track error is assumed to be identical to the errors at the first turn initiation.
- **(Position Uncertainty Model of VOR SID)** Various standards are compared to define the VOR error models in the MSG report[13]. As a result of a comparison study, the MSG report concluded that a 95% containment of VOR error was given by the following distributions. Along-track error and cross-track error is assumed to follow the double exponential distribution whose 95% containment probability is given by the following formula.

$$g \times \sin(5.2°) \qquad (20)$$

Here g is the ground distance of the aircraft from VOR. In this case, the scale parameter λ of the DE model is given by the following formula. λ =0.181 when g=6NM.

$$\lambda = g \times \sin(5.2°)/3 \qquad (21)$$

Nominal Trajectory/Uncertainty Model for Arriving Aircraft

We define the nominal trajectory and uncertainty model for arriving aircraft. The following trajectory is assumed.

- IAS = 120 knots for Cat. A aircraft and IAS = 240 knots for Cat. C aircraft during the flight.
- TAS is assumed to be identical to the TAS at 3,000ft during the flight. No wind is assumed.
- The turns conducted by RNP aircraft at waypoints are ‘fly-by’ turn.
- Actual time over waypoint for 7.8° fly-by turn is an undetermined parameter.
- RNP aircraft are nominally at 3,000ft in altitude.
- We use the Gaussian distribution model for the along/cross-track uncertainty of RNP aircraft. The standard deviation is given by the following formula.

$$\sigma = X/2.23 \qquad (22)$$

Here X denotes the 95% containment value.

4.4. Other assumptions for collision risk estimation

The following is the assumptions employed for the collision risk estimation.

Assumptions for collision risk estimation:

- **(Duration of operation)** The duration of operation in consideration for departing aircraft is from the takeoff of the departing aircraft till it arrives at VOR. For RNP aircraft, the duration of operation in consideration is from the initiation of the first turn till completion of the last turn.
- **(Average size of aircraft)** We assume that the average horizontal size of an aircraft is 0.035 NM and the average height of an aircraft is 0.010NM.

$$\lambda_{hor} = 0.035 \qquad (23)$$

$$\lambda_z = 0.010 \qquad (24)$$

- **(Relative horizontal velocity)** We assume zero speed error because the effect of speed uncertainty on the collision risk value is expected to be relatively smaller than the effect of position uncertainty. Hence the average relative horizontal velocity is given by

$$\overline{(v_{rel})_t} = \sqrt{V_{hor,1}^2(t) + V_{hor,2}^2(t) - 2V_{hor,1}(t)V_{hor,2}(t)\cos\theta(t)}, \qquad (25)$$

where $V_{hor,1}(t)$ and $V_{hor,2}(t)$ are the horizontal velocities of the departing aircraft and the RNP aircraft, respectively. The notation θ(t) denotes the angle made by the speed vectors of the two aircraft at time t.

- **(Relative vertical velocity)** The relative vertical speed is given by

$$\overline{|v_z|_t} = |(\text{climb gradient of departing aircraft (\%)}) \times V_{hor,1} - (\text{climb gradient of RNP aircraft (\%)}) \times V_{hor,2}|/100. \qquad (26)$$

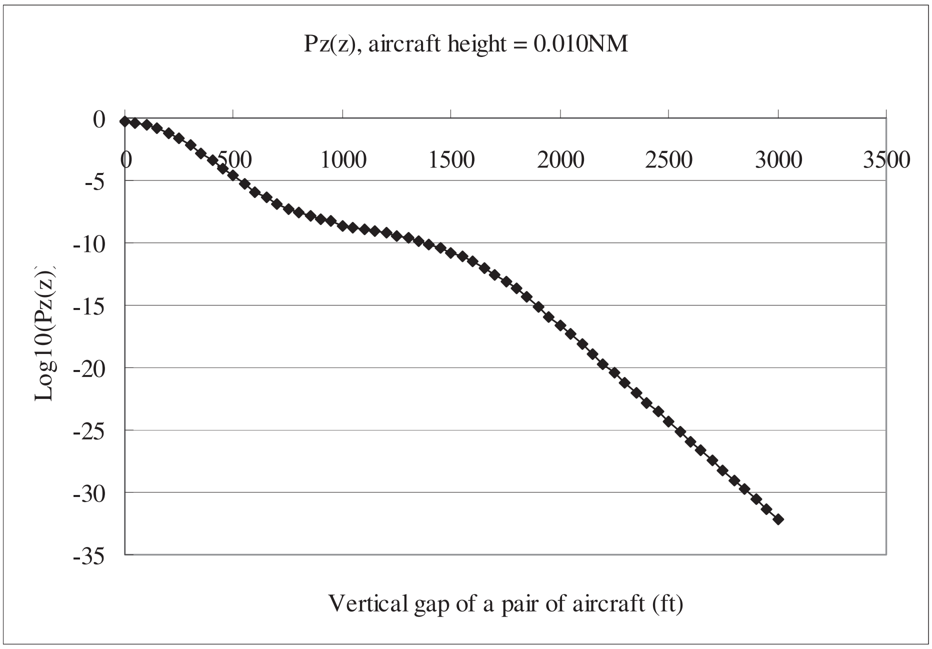

Figure 9: Vertical overlap probability

- **(Vertical overlap probability)** We use the vertical deviation model used in SASP. Figure 9 shows the graph of vertical overlap probability under this model.

4.5. Computation of horizontal overlap probability

The closed analytic form of probability density functions of distribution models $f_{a,1}(x)$, $f_{c,1}(x)$, $f_{a,2}(x)$ and $f_{c,2}(x)$ of along-track/cross-track errors of aircraft 1 and aircraft 2 are derived if the distribution models of the position error components are double exponential distribution, Gaussian distribution and the linear combination of them. We used this formula, but we omit the detail of the formula.

Consider the plane coordinate whose x-axis coincides with the ac1 flight direction. We also assume that ac1 is at the origin of the coordinate at any given time. Let (X,Y) be the coordinate of the nominal position of ac2 at any given time. Let θ be the clockwise angle made by the flight directions of ac1 and ac2. See Figure 10.

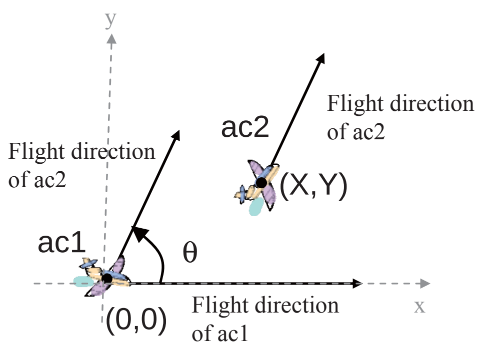

Figure 10: Horizontal overlap probability

Assume that ac2 deviates by x_2 (NM) and y_2 (NM) in the x-direction and y-direction, respectively. The deviation of the along-track direction of ac2 is $x_2 \cos\theta + y_2 \sin\theta$. The deviation of the cross-track direction of ac2 is $-x_2 \sin\theta + y_2 \cos\theta$. Then the probability density function of the relative horizontal position of ac2 in relation to ac1 is given by the following formula:

$$g(x,y) = \iint_{x_1 - x_2 = x,\, y_1 - y_2 = y} f_{a,1}(x_1) f_{c,1}(y_1) f_{a,2}(x_2 \cos\theta + y_2 \sin\theta) f_{c,2}(-x_2 \sin\theta + y_2 \cos\theta) dx_1 dy_1 \quad (27)$$

Hence, g(X,Y) is given by the following formula:

$$g(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{a,1}(x_1) f_{c,1}(y_1) f_{a,2}((x_1 - X)\cos\theta + (y_1 - Y)\sin\theta) f_{c,2}(-(x_1 - X)\sin\theta + (y_1 - Y)\cos\theta) dx_1 dy_1 \quad (28)$$

The double integral g(X,Y) is evaluated by the Gauss-Laguerre integration formula. The horizontal overlap probability $P_{hor,t}(overlap)$ is approximately given by the following formula.

$$P_{hor,t}(overlap) = \pi \lambda_h^2 g(X,Y) \quad (29)$$

4.6. Result

We apply Theorem and Proposition 2 to this example scenario. When evaluating collision risk by means of the proposed methodology, flow rates of departing and arriving traffic should be specified. In this estimation, we assumed a flow rate of 6 aircraft per hour, based on an assumption of 100,000 movements per annum. (6×2(departing flow + arriving flow)×24×365=105,210) Table 3 gives the collision risk estimations. Collision risk is expressed in the units of expected number of collisions per hour and fatal accidents per flight hour.

Table 3: Collision risk estimation

Category (Assumption: arrival and departure aircraft belonging to an identical aircraft category)	Cat. A	Cat. A	Cat. C	Cat. C
RNP-type (arrival)	RNP 0.3	RNP 1	RNP 0.3	RNP 1
Collision risk (collisions per hour)	6.28×10^{-9}	8.18×10^{-10}	1.96×10^{-6}	2.27×10^{-7}
Collision risk (fatal accidents per flight hour)	1.09×10^{-8}	1.23×10^{-9}	5.47×10^{-6}	7.11×10^{-7}

A typical procedure for the quantitative risk assessment using CRM is as follows: We first determine the safety target in advance. For instance, the value of 5.0×10^{-9} fatal accidents per flight hour has been often used for en-route separations. A procedure/separation is considered to be safe if the collision risk estimation is not larger than the pre-determined target. If we apply the same target as the en-route case into this example, we can only conclude that Cat.A versus RNP 1 case is safe. Note that this example scenario is an imaginary test case, and there is no implementation plan of this example procedure. This test case shows the applicability of our CRM for the complex scenario.

5. Conclusions

We first reviewed four collision risk models (CRMs), namely, the Reich, Hsu, Anderson and Rice CRMs. The only

Rice model can handle time-dependent position error cases and do not assume straight flight. However, the Rice CRM considers the expected number of collisions of a single pair of aircraft during the given time interval.

In the application of a collision risk model, we sometimes consider two independently-operated traffic flows. We want to know the expected total number of collisions between independently operated traffic flows rather than the expected number of collisions of a single pair of aircraft. We gave a definition of homogeneous traffic flow and proposed a CRM applicable for two independently-operated homogeneous traffic flows. This CRM is based on the philosophy of the Rice CRM and gives the total expected number of collisions between independently-operated homogeneous traffic flows. The risk exposure function of our collision risk formula is still too abstract for actual computations. Therefore, we also gave the simplified forms of the risk exposure functions in the box-shaped and cylinder-shaped collision volume cases. The position and the speed are assumed to be independent for the derivation of such simplifications in previous works. We derived the simplifications under a weakened condition which is assumed to be satisfied if the size of aircraft is significantly smaller than the scale parameters of distribution models.

We applied our CRM to the complex example scenario. In this scenario, aircraft depart from the airport using VOR, and RNP aircraft approaching the airport make the arrival traffic flow near the departure flow. The computation result showed the applicability of our CRM to the collision risk estimation of complex terminal operation scenarios.

Acknowledgement

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Appendix A: Proof of the proposition

The boundary of a box $\partial\Omega$ consists of the ceiling, the floor, the left side, the right side, the front and the back.

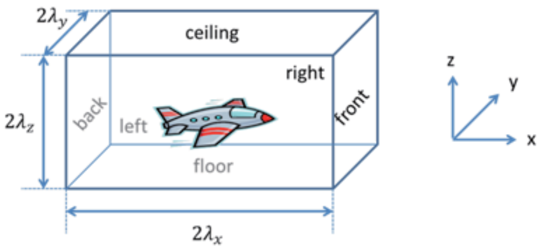

Figure 6: Ceiling, floor, front, back, left and right

It is easy to check that $\int_{\text{ceiling}}(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}+\int_{\text{floor}}(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}=4\lambda_x\lambda_y|\dot{z}_{rel}|$, $\int_{\text{left}}(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}+\int_{\text{right}}(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}=4\lambda_z\lambda_x|\dot{y}_{rel}|$ and $\int_{\text{front}}(n\cdot \mathrm{V}_{rel})^{+}dX+\int_{\text{back}}(n\cdot \mathrm{V}_{rel})^{+}dX=4\lambda_y\lambda_z|\dot{x}_{rel}|$. Therefore, the following equation holds true.

$$\int_{\partial\Omega}(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}=8\lambda_x\lambda_y\lambda_z\left(\frac{|\dot{x}_{rel}|}{2\lambda_x}+\frac{|\dot{y}_{rel}|}{2\lambda_y}+\frac{|\dot{z}_{rel}|}{2\lambda_z}\right) \tag{30}$$

Let $f_{rel,x}(s,t,\mathrm{x}_{rel},\dot{x}_{rel})=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f_{1,x}(s,\mathrm{x}_{rel}+\zeta,\dot{x}_{rel}+\eta)f_{2,x}(t,\zeta,\eta)d\zeta\, d\eta$. We also define $f_{rel,y}(s,t,\mathrm{y}_{rel},\dot{y}_{rel})$ and $f_{rel,z}(s,t,\mathrm{z}_{rel},\dot{z}_{rel})$ in the same way. By definition, $\mathrm{f}_{rel}(s,t,\mathrm{X}_{rel},\mathrm{V}_{rel})=f_{rel,x}(s,t,\mathrm{x}_{rel},\dot{x}_{rel})f_{rel,y}(s,t,\mathrm{y}_{rel},\dot{y}_{rel})f_{rel,z}(s,t,\mathrm{z}_{rel},\dot{z}_{rel})$. We evaluate $\int_{\partial\Omega}f_{rel}(s,t,X_{rel},V_{rel})(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}$. Since $\partial\Omega$ is included in a neighbourhood of $[-\lambda_x,\lambda_x]\times[-\lambda_y,\lambda_y]\times[-\lambda_z,\lambda_z]$, $\int_{\partial\Omega}f_{rel}(s,t,X_{rel},V_{rel})(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}\approx\int_{\partial\Omega}f_{rel}(s,t,0,V_{rel})(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}=f_{rel}(s,t,0,V_{rel})\int_{\partial\Omega}(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}$. Hence the following approximation holds true.

$$\begin{aligned}&\int_{\partial\Omega}f_{rel}(s,t,X_{rel},V_{rel})(n\cdot \mathrm{V}_{rel})^{+}dX_{rel}\\&=8\lambda_x\lambda_y\lambda_z f_x(s,t,0,\dot{x}_{rel})f_y(s,t,0,\dot{y}_{rel})f_z(s,t,0,\dot{z}_{rel})\left(\frac{|\dot{x}_{rel}|}{2\lambda_x}+\frac{|\dot{y}_{rel}|}{2\lambda_y}+\frac{|\dot{z}_{rel}|}{2\lambda_z}\right)\end{aligned} \tag{31}$$

Since $\overline{|\dot{x}_{rel}|}_{s,t}$ is the average relative longitudinal speed of a pair of colliding aircraft, especially being a conditional expectation, $\overline{|\dot{x}_{rel}|}_{s,t}$ is given by the following equation.

$$\overline{|\dot{x}_{rel}|}_{s,t} = \frac{\int_{-\infty}^{+\infty}\int_{-\lambda_x}^{\lambda_x} |\dot{x}_{rel}| f_{rel,x}(s,t,\mathrm{x}_{rel},\dot{x}_{rel}) d\mathrm{x}_{rel}\, d\dot{x}_{rel}}{\int_{-\infty}^{+\infty}\int_{-\lambda_x}^{\lambda_x} f_{rel,x}(s,t,\mathrm{x}_{rel},\dot{x}_{rel}) d\mathrm{x}_{rel}\, d\dot{x}_{rel}}$$

$$\approx \frac{2\lambda_x \int_{-\infty}^{+\infty} |\dot{x}| f_{rel,x}(s,t,0,\dot{x}_{rel}) d\dot{x}_{rel}}{2\lambda_x \int_{-\infty}^{+\infty} f_{rel,x}(s,t,0,\dot{x}_{rel}) d\dot{x}_{rel}} \tag{32}$$

$$= \frac{\int_{-\infty}^{+\infty} |\dot{x}| f_{rel,x}(s,t,0,\dot{x}_{rel}) d\dot{x}_{rel}}{\int_{-\infty}^{+\infty} f_{rel,x}(s,t,0,\dot{x}_{rel}) d\dot{x}_{rel}}$$

We used the assumption that $f_{rel,x}(s,t,\mathrm{x}_{rel},\dot{x}_{rel})$ is approximated by $f_{rel,x}(s,t,0,\dot{x}_{rel})$ on a neighbourhood of $\mathbb{R}\times\mathbb{R}\times[-\lambda_x,\lambda_x]\times\mathbb{R}$. We can obtain the approximations $\overline{|\dot{y}_{rel}|}_{s,t} \approx \int_{-\infty}^{+\infty} |\dot{y}_{rel}| f_{rel,y}(s,t,0,\dot{y}_{rel}) d\dot{y}_{rel} / \int_{-\infty}^{+\infty} f_{rel,y}(s,t,0,\dot{y}_{rel}) d\dot{y}_{rel}$ and $\overline{|\dot{z}_{rel}|}_{s,t} \approx \int_{-\infty}^{+\infty} |\dot{z}_{rel}| f_{rel,z}(s,t,0,\dot{z}_{rel}) d\dot{z}_{rel} / \int_{-\infty}^{+\infty} f_{rel,z}(s,t,0,\dot{z}_{rel}) d\dot{z}_{rel}$.

As to the overlap probability,

$P_{x,s,t}(overlap) = \int_{-\infty}^{+\infty}\int_{-\lambda_x}^{\lambda_x} f_{rel,x}(s,t,\mathrm{x}_{rel},\dot{x}_{rel}) d\mathrm{x}_{rel}\, d\dot{x}_{rel} \approx 2\lambda_x \int_{-\infty}^{+\infty} f_{rel,x}(s,t,0,\dot{x}_{rel}) d\dot{x}_{rel}$, $P_{y,s,t}(overlap) \approx 2\lambda_y \int_{-\infty}^{+\infty} f_{rel,y}(s,t,0,\dot{y}_{rel}) d\dot{y}_{rel}$ and $P_{z,s,t}(overlap) \approx 2\lambda_z \int_{-\infty}^{+\infty} f_{rel,z}(s,t,0,\dot{z}_{rel}) d\dot{z}_{rel}$.

We obtain equation (10) by integrating equation (13) over $\mathbb{R}^3$ and using the above equations.