[EN-A-037] Estimation of 3-dimensional Location and Velocity

Using Range with Bias Error and Doppler Measurements

(EIWAC 2017)

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Abstract: This paper deals with a target location and velocity estimation algorithm in a TOA (Time of Arrival) system by Taylor-series estimation. In this paper, we assume that each range measurement has a bias error due to the clock offset or multipath propagation effect. We present two methods. The first method is to estimate the target location, velocity and each range bias using n pairs of range with bias error and Doppler measurements. The second method is to estimate the target location and velocity using only n Doppler measurements. We present observability condition for estimating the target location and velocity using these methods. We illustrate that there is not difference between them. We also provide measures of location and velocity estimation accuracy.

Keywords: TOA, GPS, location system, error analysis, range, range bias error, Doppler

1. INTRODUCTION

This paper deals with a target location and velocity estimation algorithm in a TOA (Time of Arrival) system such as the GPS (Global Positioning System).

The TOA measurements between the reference point at known location and the target are multiplied by the light speed in order to obtain range measurements. Then, a target location can be estimated from several range measurements between them as shown in Fig. 1 [1]–[5].

Here, the range measurement is a non-linear function of 3-dimensional target location. We use Taylor-series estimation (also Gauss-Newton algorithm) that is an iterative method for solving non-linear least squares problems, starting with an initial estimate and converging the estimate [5]. However, accuracy of estimated target location is very sensitive to target and reference points geometry.

In the conventional TOA system, it was assumed that the clocks of all reference points are synchronized and there exists a clock offset between the transmitter clock and the receiver clock [1]–[4]. In this case, it is necessary to obtain at least four range measurements to estimate 3-dimensional target location and the clock offset [1]–[4].

In this paper, we assume that each range measurement has a bias error due to the clock offset or multipath propagation effect. If we estimate each range bias error and 3-dimensional target location from n reference points, we must solve a linear problem with n+3 unknowns and n equations. However, we cannot solve this problem because this is the undetermined case.

By the way, we have developed the method of estimating 3-dimensional location and velocity using several range and Doppler measurements [6]. We proved that we can always estimate 3-dimensional target location and velocity using several range and Doppler measurements when we can estimate 3-dimensional target location using the conventional TOA system with only range measurements [6]. We also proved that estimated location accuracy becomes better when using additional Doppler measurements even in cases of poor target and reference points geometry or bad Doppler measurement accuracy [6].
The problem of estimating target location in the multipath propagation environments has been reported. But they assume the existence of some range measurements without bias error [7] or knowledge of the multipath propagation error statics [8],[9].

The problem of estimating target location with Doppler measurements has been also reported. But they do not estimate target velocity [10].

In this paper, we present two methods to estimate the target location and velocity using additional Doppler measurements.

The first method is to estimate the target location, velocity and each range bias using n pairs of range with bias error and Doppler measurements. We call this method a BE (Bias Estimation) method. In this case, we must solve a linear problem with n+6 unknowns and 2n equations. Therefore, at least 6 pairs of measurements must be used.

The second method is to estimate the target location and velocity using only n Doppler measurements. We call this method a DO (Doppler Only) method. In this case, we must solve a linear problem with 6 unknowns and n equations. Therefore, at least 6 measurements must be used.

We present the observability condition for estimating the target location and velocity using these methods. We compare the BE method with the DO method analytically. We also provide measures of location and velocity estimation accuracy. This measure is the eigenvalue of a matrix calculated from reference points and the initial (nominal) value of the target’s location and velocity.

2. MEASUREMENT MODEL

In this chapter, we assume that we can obtain n pairs of range and Doppler measurements from n reference points.

2.1 Range Measurement Model

Let \( D^T \) be the transpose of a matrix \( D \) and \( B \), be a known location vector of an \( i(i=1,2,\ldots,n) \)th reference point in 3-dimensional Cartesian coordinates given by:

\[
B = (x_i, y_i, z_i)^T
\]

Let \( L \) be an unknown location vector of the target in 3-dimensional Cartesian coordinates given by:

\[
L = (x, y, z)^T
\]

Then, the true range (distance) \( R \) between the target location and the \( i \)th reference point is defined as:

\[
R_i = f_i(L)
\]

where

\[
f_i(L) = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}
\]

Therefore, the range measurement \( R_m \) between a target location and the \( i \)th reference point can be obtained as:

\[
R_m = R_i + b_i + v_i
\]

where \( b_i \) is the range bias error by the clock offset or multipath propagation effect, \( v_i \) is the random range measurement noise.

Then, we obtain the following property using the total differential (Taylor-series linearization).

(Property1) Let \( L_0 = (x_0, y_0, z_0)^T \) be an initial (nominal) estimate of the target location. Then the following result holds:

\[
\Delta R_m = \left( \begin{array}{ccc}
\alpha_i & \beta_i & \gamma_i \\
\end{array} \right) \Delta L + b_i + v_i
\]

where

\[
\Delta R_m = R_m - f_i(L_0)
\]

\[
\Delta L = L - L_0
\]

\[
\alpha_i = \frac{x_0 - x_i}{f_i(L_0)}, \beta_i = \frac{y_0 - y_i}{f_i(L_0)}, \gamma_i = \frac{z_0 - z_i}{f_i(L_0)}
\]

Here, \((\alpha_i, \beta_i, \gamma_i)\) is the unit vector from the reference point \( i \) to the initial target location \( L_0 \), as shown in Fig. 2.

Let \( J \) be an \( n \times n \) unit matrix. The following linear model can be obtained from property1 when n reference points are used:

\[
\tilde{z}_i = \left( A_i \begin{array}{c}
J_n \end{array} \right) \left( \begin{array}{c}
\alpha_i \\
\beta_i \\
\gamma_i \\
\end{array} \right) + v_i
\]

where we define the following equation using Eq. (9)

\[
\omega(i) = \left( \begin{array}{ccc}
\alpha_i & \beta_i & \gamma_i \\
\end{array} \right)
\]

and

\[
A_i = \left( \omega(1)^T \cdots \omega(n)^T \right)^T
\]
\[ \mathbf{z}_d = (\Delta R_{10}, \Delta R_{20}, \ldots, \Delta R_{m0})^T \]  
\[ \mathbf{a}_b = (b_1, b_2, \ldots, b_n)^T \]  
\[ \mathbf{v}_d = (v_1, v_2, \ldots, v_n)^T \]  

Here, \( A_i \) is an \( n \times 3 \) matrix.

### 2.2 Doppler Measurement Model

Let \( \hat{\mathbf{b}} \) be a known velocity vector of an \( i (i = 1, 2, \ldots, n) \)th reference point in 3-dimensional Cartesian coordinates given by:

\[ \hat{\mathbf{b}} = (\hat{x}_i, \hat{y}_i, \hat{z}_i)^T \]  

Let \( \mathbf{L}_0 \) be an unknown velocity vector of the target in 3-dimensional Cartesian coordinates given by:

\[ \mathbf{L}_0 = (x_0, y_0, z_0)^T \]

Then, the true Doppler \( \hat{\mathbf{r}}_i \) between the target location and the \( i \)th reference point is defined as:

\[ \hat{\mathbf{r}}_i = g_i(L, \mathbf{L}) \]  

where

\[ g_i(L, \mathbf{L}) = \frac{(\mathbf{L} - L_0)^T (\mathbf{L} - L_0)}{f_i(L)} \]  

Therefore, the Doppler measurement \( \hat{\mathbf{r}}_i \) between a target location and the \( i \)th reference point can be obtained as:

\[ \hat{\mathbf{r}}_i = \hat{\mathbf{r}}_i + \mathbf{v}_i \]  

where \( \mathbf{v}_i \) is the random Doppler measurement noise.

Then, we obtain the following property using the total differential (Taylor-series linearization) using Eq. (11).

**Property 2** Let \( \mathbf{L}_0 = (x_0, y_0, z_0)^T \) be an initial (nominal) estimate of the target velocity. Then the following result holds:

\[ \Delta \mathbf{R}_i = \begin{pmatrix} \alpha \alpha \beta \beta \end{pmatrix} \mathbf{a}_i + \mathbf{a}_i \mathbf{a}_i + \dot{\mathbf{v}}_i \]  

where

\[ \Delta \mathbf{R}_i = \hat{\mathbf{r}}_i - g_i(L_0, \mathbf{L}_0) \]

\[ \alpha = \frac{\hat{x}_0 - \hat{x}_i}{f_i(L_0)} \]

\[ \beta = \frac{\hat{y}_0 - \hat{y}_i}{f_i(L_0)} \]

\[ \gamma = \frac{\hat{z}_0 - \hat{z}_i}{f_i(L_0)} \]  

The following linear model can be obtained from property 2, Eqs. (11) and (12) when \( n \) reference points are used:

\[ \mathbf{z}_d = B \mathbf{b} + \mathbf{v}_d \]

where we define the following equation using Eq. (23)

\[ \boldsymbol{\kappa}(i) = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \]  

and

\[ A_{id} = \begin{pmatrix} \boldsymbol{\kappa}(1)^T & \cdots & \boldsymbol{\kappa}(n)^T \end{pmatrix}^T \]

\[ B = \begin{pmatrix} A_{id} \ A_i \end{pmatrix} \]

\[ \mathbf{z}_d = (\Delta \mathbf{R}_{10}, \Delta \mathbf{R}_{20}, \ldots, \Delta \mathbf{R}_{m0})^T \]

\[ \mathbf{b} = \begin{pmatrix} (\mathbf{a}_0^T \cdot \mathbf{b})^T \end{pmatrix} \]

\[ \mathbf{v}_d = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)^T \]  

Here, \( A_{id} \) is an \( n \times 3 \) matrix and \( B \) is an \( n \times 6 \) matrix.

### 2.3 Linear Model of Measurements

Let \( O_{n \times n} \) be an \( m \times n \) zero matrix. The following linear model can be obtained from property 1 and property 2 when \( n \) reference points are used:

\[ \mathbf{z} = A \mathbf{a} + \mathbf{v} \]

where

\[ \mathbf{z} = \begin{pmatrix} \mathbf{z}_0^T, \mathbf{z}_d^T \end{pmatrix}^T \]

\[ A = \begin{pmatrix} A_i & O_{n,3} & \mathbf{I}_n \end{pmatrix} \]

\[ \mathbf{a} = \begin{pmatrix} (\mathbf{a}_0^T, \mathbf{a}_1^T, \mathbf{a}_n^T) \end{pmatrix} \]

\[ \mathbf{v} = \begin{pmatrix} (\mathbf{v}_0, \mathbf{v}_d) \end{pmatrix}^T \]  

and we call \( A \) a placement matrix. Here, \( A \) is a \( 2n \times (n+6) \) matrix.

### 2.4 Measurement Noise Covariance Matrix

We assume that measurement noises are zero mean and measurement noise covariance matrix is positive definite as follows:

\[ E[\mathbf{v}] = \mathbf{0} \]  

(37)
$$V = E\left[ v v^T \right] = \begin{pmatrix} V_v & V_d \\ V_d^T & V_d \end{pmatrix} > 0 \quad (38)$$

where

$$V_v = E\left[ v v^T \right] \quad (39)$$

$$V_d = E\left[ v_d v_d^T \right] \quad (40)$$

$$V_d = E\left[ v_d v_d^T \right] \quad (41)$$

Here, $E\left[ \cdot \right]$ indicates the mean, $\hat{0}$ indicates the zero vector, $D > 0$ indicates that a matrix $D$ is positive definite, and $D \geq 0$ indicates that a matrix $D$ is positive semi-definite.

### 3. ESTIMATION METHOD

In this chapter, we present two methods to estimate the target location and velocity using range with bias error and Doppler measurements.

#### 3.1 DO (Doppler Only) method

The following property shows that we can estimate the target location and velocity only using $n$ Doppler measurements by the weighted least-squares method [11], [12]. Namely, we minimize the following equation to obtain an optimal estimate $\hat{b}$ for the location and velocity vector:

$$J = (\hat{z} - A\hat{a})^T V_{1}^{-1} (\hat{z} - A\hat{a}) \quad (46)$$

Then, we obtain the following property.

**Property5** Let $A^T V_{1}^{-1} A$ be invertible. Then the following result holds:

$$\hat{a} = (A^T V_{1}^{-1} A)^{-1} A^T V_{1}^{-1} \hat{z} \quad (47)$$

The following property shows that $\hat{a}$ is the unbiased estimate and its estimated covariance matrix [11], [12].

**Property6** The following results hold:

$$E[\hat{a}] = a \quad (48)$$

$$E[(\hat{a} - a)(\hat{a} - a)^T] = (A^T V_{1}^{-1} A)^{-1} \quad (49)$$

### 4. ANALYSIS

In this chapter, we present the observability condition. We also compare the estimated accuracy between the DO method only using Doppler measurements and the BE method using range and Doppler measurements theoretically.

#### 4.1 Prerequisite

In this section, we propose some prerequisites for estimating the target location and velocity.

**Prerequisite1** The rank of a matrix $B$ is 6.

**Prerequisite2** The following result holds:

$$B^T B > 0 \quad (50)$$

**Prerequisite3** Either of the following results holds:

$$A_i^T A_i > 0 \quad \text{and} \quad A_i^T A_{ui} > A_i^T A_i (A_i^T A_i)^{-1} A_i^T A_{ui} \quad (51)$$

$$A_{ui}^T A_{ui} > 0 \quad \text{and} \quad A_i^T A_i > A_i^T A_i (A_i^T A_i)^{-1} A_{ui}^T A_{ui} \quad (52)$$

**Prerequisite4** The rank of a matrix $A$ is $n+6$.

**Prerequisite5** The following result holds:

$$A^T A > 0 \quad (53)$$

We illustrate the relations among the prerequisites mentioned above.

**Property7** The prerequisite1 and 2 are equivalent.

**Proof** The proof can be obtained by the property of the rank and the positive definite matrix.

**Property8** The prerequisite2 and 3 are equivalent.

**Proof** From Eq. (28), the result is
\[
B^T B = \begin{pmatrix}
A^T u A_u & A^T u A_t \\
A^T u A_d & A^T v A_l
\end{pmatrix}
\] (54)

Here, \(B^T B\) is positive semi-definite matrix. Therefore, the proof can be obtained by the Lemma 1 in the appendix.

(Property 9) The prerequisite 4 and 5 are equivalent.
(Proof) The proof can be obtained by the property of the rank and the positive definite matrix.

(Property 10) The prerequisite 1 and 4 are equivalent.
(Proof) The proof can be obtained by the property of the rank.

From the property 7–10, we can conclude the following result.
(Property 11) The prerequisite 1–5 are all equivalent.

4.2 Observability Condition
The following property shows the observability condition of the DO method using \(n\) Doppler measurements.
(Property 12) The prerequisite 1 holds if and only if the following result holds:
\[
B^T V^{-1}_d B > 0
\] (55)
(Proof) From Eq. (38), the result is
\[
V_d > 0
\] (56)
Therefore, the proof can be obtained by the property of the rank and the positive definite matrix.

The following property shows the observability condition of the BE method using \(n\) pairs of Doppler and range measurements.
(Property 13) The prerequisite 4 holds if and only if the following result holds:
\[
A^T V^{-1} A > 0
\] (57)
(Proof) The proof can be obtained by the property of the rank and the positive definite matrix using Eq. (38).

From the property 11–13, we can conclude the following result.
(Property 14) The prerequisite 1–5, \(B^T V^{-1}_d B\) has an inverse, and \(A^T V^{-1} A\) has an inverse are all equivalent.

From the property 14, Eqs. (43) and (47), the prerequisite 1–5 are observability condition for estimating target location and velocity.

4.3 Comparison
In this section, we compare the estimated location and velocity of the target between the DO method and the BE method.

The following equation can be obtained from Eqs. (30) and (35)
\[
\hat{h} = N_a \hat{a}
\] (58)
where
\[
N_a = \begin{pmatrix}
I_n & O_{n,s}
\end{pmatrix}
\] (59)

Then the following equation and Eq. (47) show the estimated location and velocity in the BE method without estimated range bias.
\[
\hat{a}_v = N_a \hat{a}
\] (60)
The following property shows that \(\hat{a}_v\) is the unbiased estimate and its estimated covariance matrix.
(Property 15) The following results hold:
\[
E[\hat{a}_v] = \bar{h}
\] (61)
\[
E[(\hat{a}_v - \bar{h})(\hat{a}_v - \bar{h})^T] = N_a\left( A^T V^{-1} A \right)^{-1} N_a^T
\] (62)
(Proof) The proof can be obtained by the property 6, Eqs. (58) and (60).

Therefore, it is necessary to evaluate \(\hat{a}_v\) and \(\hat{b}\) to compare the estimated location and velocity.

The following Eq. (63) shows that error characteristics are the same for the DO method and the BE method from Eqs. (45) and (62). Furthermore, the following Eq. (64) shows that there is no difference between the two methods from Eq. (43).
(Property 16) Let one of prerequisite 1–5 hold. Then the following results hold:
\[
N_a \left( A^T V^{-1} A \right)^{-1} N_v^T = (B^T V_d^{-1} B)^{-1}
\] (63)
\[
\hat{a}_v = \hat{b}
\] (64)
(Proof) From Eq. (28), the results are
\[
B^T V_d^{-1} = \begin{pmatrix}
A^T v V^{-1}_d A_d & A^T v V^{-1}_d A_l \\
A^T v V^{-1}_d A_d & A^T v V^{-1}_d A_l
\end{pmatrix}
\] (65)
\[
B^T V_d^{-1} B = \begin{pmatrix}
A^T v V^{-1}_d A_d & A^T v V^{-1}_d A_l \\
A^T v V^{-1}_d A_d & A^T v V^{-1}_d A_l
\end{pmatrix}
\] (66)
From Eqs. (43) and (65), the result is
\[
\hat{\theta} = (B^T V_d^{-1} B)^{-1} \begin{pmatrix}
A_\alpha^T V_d^{-1} \xi_d \\
A_\beta^T V_d^{-1} \xi_d
\end{pmatrix}
\]  
(67)

We can define the following relation from Eq. (38):
\[
V^{-1} = \begin{pmatrix}
G_{11} & G_{12} \\
G_{12} & G_{22}
\end{pmatrix} > 0
\]  
(68)

where \(G_{11}, G_{12}, G_{22}\) are \(n \times n\) matrices.

From Eqs. (68) and (38) using the Lemma1 in the appendix, the result is
\[
V_d^{-1} = G_{22} - G_{12} G_{11}^{-1} G_{12}
\]  
(69)

From Eqs. (34) and (68), the results are
\[
A^T V^{-1} = \begin{pmatrix}
A_\alpha^T G_{11} + A_\alpha^T G_{12}^T & A_\alpha^T G_{12} + A_\alpha^T G_{22} \\
A_\beta^T G_{11}^T & A_\beta^T G_{22}
\end{pmatrix}
\]  
(70)

\[
A^T V^{-1} A = \begin{pmatrix}
K_{11} & K_{12} \\
K_{12}^T & K_{11}
\end{pmatrix}
\]  
(71)

where
\[
P = A_\alpha^T G_{11} A_\alpha + A_\alpha^T G_{12} A_\beta + A_\beta^T G_{12} A_\alpha + A_\beta^T G_{22} A_\beta
\]  
(72)

and
\[
K_{11} = \begin{pmatrix}
P + A_\alpha^T G_{22} A_\alpha & A_\alpha^T G_{12} A_\beta + A_\alpha^T G_{22} A_\beta \\
A_\beta^T G_{12} A_\alpha + A_\beta^T G_{22} A_\beta & A_\beta^T G_{22}
\end{pmatrix}
\]  
(73)

\[
K_{12} = \begin{pmatrix}
A_\alpha^T G_{11} & A_\alpha^T G_{12} \\
A_\beta^T G_{12}^T & A_\beta^T G_{22}
\end{pmatrix}
\]  
(74)

From Eq. (71) using the Lemma1 in the appendix and Eq. (59), the result is
\[
N_1 (A^T V^{-1} A)^{-1} N_1^T = K_{11} - K_{12} G_{11}^{-1} K_{12}^T
\]  
(75)

From Eq. (74) using Eq. (72), the result is
\[
K_{12} G_{11}^{-1} K_{12}^T = \begin{pmatrix}
P + A_\alpha^T F A_\alpha & A_\alpha^T G_{12} A_\beta + A_\alpha^T F A_\beta \\
A_\beta^T G_{12} A_\alpha + A_\beta^T F A_\alpha & A_\beta^T F A_\beta
\end{pmatrix}
\]  
(76)

where
\[
F = G_{11} G_{12}^{-1} G_{22}
\]  
(77)

From Eqs. (69) and (77), the result is
\[
V_d^{-1} = G_{22} - F
\]  
(78)

From Eqs. (73) and (76) using Eq. (78), the result is
\[
K_{11} - K_{12} G_{11}^{-1} K_{12}^T = \begin{pmatrix}
A_\alpha^T V_d^{-1} A_\alpha & A_\alpha^T V_d^{-1} A_\beta \\
A_\beta^T V_d^{-1} A_\alpha & A_\beta^T V_d^{-1} A_\beta
\end{pmatrix}
\]  
(79)

Eq. (63) has been proved from Eqs. (75) and (79) using Eq. (66).

By the way, from Eqs. (71) and (59) using the Lemma1 in the appendix and Eq. (63), the result is
\[
N_1 (A^T V^{-1} A)^{-1} = \begin{pmatrix}
(B^T V_d^{-1} B)^{-1} & -(B^T V_d^{-1} B)^{-1} K_{12} G_{11}^{-1}
\end{pmatrix}
\]  
(80)

From Eqs. (80) and (70), the result is
\[
N_1 (A^T V^{-1} A)^{-1} A^T V^{-1} = \begin{pmatrix}
A^T G_{11} + A^T G_{12}^T & A^T G_{12} + A^T G_{22} \\
A^T G_{12}^T & A^T G_{22}
\end{pmatrix}
\]  
(81)

From Eq. (81) using Eq. (74), the result is
\[
N_1 (A^T V^{-1} A)^{-1} A^T V^{-1} = \begin{pmatrix}
O_{3, n} & A^T G_{22} - G_{12} G_{11}^{-1} G_{12}
\end{pmatrix}
\]  
(82)

From Eq. (82) using Eq. (69), the result is
\[
N_1 (A^T V^{-1} A)^{-1} A^T V^{-1} = \begin{pmatrix}
O_{3, n} & A^T V_d^{-1} \\
O_{3, n} & A^T V_d^{-1}
\end{pmatrix}
\]  
(83)

From Eqs. (83) and (33), the result is
\[
N_1 (A^T V^{-1} A)^{-1} A^T V^{-1} Z = \begin{pmatrix}
O_{3, n} & A^T V_d^{-1} Z_d \\
O_{3, n} & A^T V_d^{-1} Z_d
\end{pmatrix}
\]  
(84)

Eq. (64) has been proved from Eqs. (60) and (47) using Eqs. (84) and (67).

### 4.4 Upper and Lower Bounds for the Estimated Errors

In this section, we illustrate the measures of location and velocity estimation accuracy independent of the measurement noise.

The following equations can be obtained from Eq. (30).
\[
a_1 = N_1 b
\]  
(85)
\[
a_2 = N_1 b
\]  
(86)

where
From Eq. (56), the result is

\[ 0 < \sigma_{\lambda,\text{min}}^2 I_s \leq \sigma_{\lambda,\text{min}}^2 I_s \]

(97)

From Eq. (97), the result is

\[ 0 < \sigma_{\lambda,\text{min}}^2 I_s \leq \sigma_{\lambda,\text{min}}^2 I_s \]

(98)

From Eq. (98), the result is

\[ \frac{1}{\sigma_{\lambda,\text{min}}^2} \leq \frac{1}{\sigma_{\lambda,\text{min}}^2} I_s \]

(99)

From Eq. (99), the result is

\[ \frac{1}{\sigma_{\lambda,\text{min}}^2} I_s \leq \frac{1}{\sigma_{\lambda,\text{max}}^2} I_s \]

(100)

From Eq. (100) using Eq. (50), the result is

\[ \sigma_{\lambda,\text{min}}^2 \leq \sigma_{\lambda,\text{max}}^2 \]

(101)

From Eq. (101), the result is

\[ \sigma_{\lambda,\text{min}}^2 \leq \sigma_{\lambda,\text{max}}^2 \]

(102)

From Eq. (102) using Eq. (96), the result is

\[ \sigma_{\lambda,\text{min}}^2 \leq \sigma_{\lambda,\text{max}}^2 \]

(103)

Eq. (91) has been proved from Eq. (103) using Eq. (93).

By the way, from the property4 using Eq. (90), the result is

\[ E[\dot{b}_l - E[\dot{b}_l]] = N_i (B^T V_i^{-1} B)^{-1} N_i^T \]

(104)

From Eq. (54) using the Lemmal 1 in the appendix, Eqs. (88) and (52), the result is

\[ N_i (B^T V_i^{-1} B)^{-1} N_i^T \]

(105)

Then, Eq. (92) has been proved from Eqs. (104) and (105) similar to the proof of Eq. (91).

5. CONSIDERATIONS

The prerequisite1 shows that at least 6 reference points must be used to estimate target location and velocity.

Furthermore, the property17 shows that we can compute \( \dot{b}_l \) and \( \dot{b}_l \) without divergence when the minimum eigenvalues \( \lambda_{\text{min}} \) and \( \lambda_{\text{min}} \) are large.

Here, it is well known that the eigenvalue of a matrix is a continuous function of elements of the matrix, but the rank of a matrix is not. Therefore, the minimum eigenvalues \( \lambda_{\text{min}} \) and \( \lambda_{\text{min}} \) can be expected as the better measure for unique solution and the estimated accuracy.
6. CONCLUSION

In this paper, we illustrated that we can estimate 3-dimensional target location and velocity using range and Doppler measurements even when each range measurement has a bias error in the TOA location system. Namely, we presented two methods, the DO method and the BE method to estimate target location and velocity. We also showed that there is not difference between both methods in terms of estimated location and velocity of the target. We presented observability condition for estimating the target location and velocity by these methods using the placement matrix. We also illustrated the measures of location and velocity estimation accuracy using the minimum eigenvalue calculated from the placement matrix.

7. REFERENCES


8. APPENDIX

(Lemma1) Let $D,D_{1}$ and $D_{2}$ be $(m+n)\times(m+n)$, $m\times m$ and $n\times n$ matrices, respectively. We assume that the following relation holds.

$$D = \begin{pmatrix} D_{1} & D_{12} \\ D_{12}^T & D_{22} \end{pmatrix} \geq 0$$

If $D > 0$, then $D_{22} > 0$ and $H = D_{11} - D_{12}D_{22}^{-1}D_{12}^T > 0$ and the following equation holds.

$$D^{-1} = \begin{pmatrix} H^{-1} & -H^{-1}D_{12}D_{22}^{-1} \\ -D_{12}^T H^{-1} & D_{22}^{-1} + D_{12}^T D_{22}^{-1}H^{-1}D_{12}D_{22}^{-1} \end{pmatrix}$$

Conversely, if $D_{22} > 0$ and $H = D_{11} - D_{12}D_{22}^{-1}D_{12}^T > 0$, then $D > 0$ and the Eq. (a2) holds.

If $D > 0$, then $D_{11} > 0$ and $L = D_{22} - D_{12}^T D_{22}^{-1}D_{12} > 0$ and the following equation holds.

$$D^{-1} = \begin{pmatrix} D_{11}^{-1} + D_{12}^T D_{22}^{-1}L^{-1}D_{12} - D_{12}^T D_{22}^{-1}L^{-1} \\ -L^{-1}D_{12}^T D_{22}^{-1}L^{-1} \end{pmatrix}$$

Conversely, if $D_{11} > 0$ and $L = D_{22} - D_{12}^T D_{22}^{-1}D_{12} > 0$, then $D > 0$ and the Eq. (a3) holds.

(Proof) Let $M \geq 0$. Then $M > 0$ if and only if $M$ has an inverse. Therefore the results have been proved by the formula of a block matrix.